NOTES ON MATHEMATICS

Mathematics provides a language in which we can express the relationships that arise in a quantitative area of science such as astronomy. With it, we can be both more concise and more precise. In theoretical branches of the subject, it is possible to work out formulae that help us to understand the underlying physics of a situation, such as the conditions in an astronomical object.

In the Certificate course we try to keep the use of mathematics to a minimum and you should be able to understand most of the material quite well even if you 'bleep over' equations when they arise. Where we do use equations, they are there to help you to understand the astronomy more fully.

In some of the homework, simple calculations are to be done, generally by substituting numbers into equations. We set these because it has been found that by getting hands-on experience of such calculations you will have a better feel for the physics and astronomy of what is going on. Some of the practical experiments also require simple calculation.

In the written examinations that come at the end of the year there may be a very few questions involving calculation. If you are comfortable in doing these they can give a quick way to accumulate good marks. If not, however, there is sufficient choice in the questions that the calculations can easily be avoided.

These notes are intended to give you a reminder of some of the basic properties of mathematical equations and how they can be used in a simple way. If you should have any difficulty with the maths in the course (or indeed with any aspect of it), then do please ask for help.

Contents

1. Introduction to Algebra

Algebra is a system of calculation where the numbers are represented by symbols. This has the advantages that general expressions can be written down, true for all numerical values of the symbols, and that manipulation can lead to other results—the calculation is done once for all rather than separately each time the numbers are changed.

Generally, each quantity is represented by one symbol, normally a letter of the alphabet a, b, etc, supplemented by Greek letters α, β, etc. [You will find a list of Greek letters and their names on the Certificate web site. These two are *alpha* and *beta*]. Both lower-case and upper-case letters can be used, and normally they have different meanings, so we must be careful to be consistent and always to write each letter the same way. It is wrong, for instance, to use upper-case M and lower-case m as though they are interchangeable, or R and r, and so on.

In some applications, particular two-letter combinations can be defined as a single quantity. For example, in calculus (a collection of techniques *not* required in the Certificate course) the combination dx is taken to represent a very small change in some quantity x. In the same way, sometimes δx or Δx [these are lower-case and upper-case Greek *delta*, respectively] might be used for a not-quite-so-small change in the value of x.

In physics, generally a symbol is used to represent some physical quantity. For example, d might represent a distance, r a radius or distance from a centre, A an area, and so on. But other letters could be used for any of these quantities, and these letters could be used elsewhere with other meanings. In any piece of algebra, the meanings attached to each symbol used should be made clear, either from the context or by actually defining them all.

In physics, some quantities can be considered as *variables*, such as a radial distance r, and others as *constants*. There are two types of constant, mathematical constants which are pure numbers and physical constants which have to do with the real world. An example of a mathematical constant is the ratio of the circumference of a circle to its diameter, conventionally written as π [lower case *pi*] and with a value 3.14159... that is given on any pocket calculator. Physical constants include all sorts of things that can be measured such as the mass of the Sun $M_\odot$ or the gravitational constant G.

2. Combining quantities

Addition and Subtraction

If we want to add two quantities a and b we write the sum as $a + b$. If we want to subtract them, we write $a - b$.

For example, if $a = 4$ and $b = 3$ then

$$a + b = 4 + 3 = 7,$$

$$a - b = 4 - 3 = 1.$$

There can be more than two items in the list. For example, if we also have $c = 7$ and $d = 6$ then

$$a - b - c + d = 4 - 3 - 7 + 6 = 0.$$

In this case the result is zero. In other cases, some of the quantities or the final result could be negative numbers. So, we could have $e = -6$, $f = -2$ and

$$d - a + e - f = 6 - 4 - 6 - (-2) = -4 + 2 = -2.$$

Note that subtracting a negative number is equivalent to adding the corresponding positive number. Note also that if we have to multiply two minus signs it is clearer if we use brackets, writing $-(-2)$ rather than $- - 2$.

Multiplication

For numbers, the multiplication sign $\times$ is used, as in $3 \times 4 = 12$. In algebra, if two symbols are written together, as ab, then they are to be multiplied. The sign $\times$ also can be used, as $a \times b$, and so can a dot between the symbols, as $a \cdot b$. So, with $a = 4$ and $b = 3$ as before,

$$\begin{aligned} ab &= a \times b = a \cdot b \\ &= 3 \times 4 = 12. \end{aligned}$$

There is no rule about which way to write it in a particular situation, except that it's often best to keep things looking simple which favours the use of ab.

With numbers, the dot notation should not be used, to avoid confusion with a decimal point in an expression such as 2.5.

Division

To divide two numbers, we can use the division sign $\div$. Thus, $4 \div 2 = 2$. The sign $\div$ can be used in algebra, as $a \div b$, but it is more common to use a line. In a display equation it can be written as

$$\frac{a}{b}$$

and to save space when writing a formula in the middle of text the forward slash or solidus / can be used, as a/b. This is consistent with the ways in which we can write a fraction, $\frac{1}{2} = 1/2$.

Brackets

Brackets give a convenient way to write things more simply. For example, if we have the combination $bc + bd$ it is simpler and generally involves less writing to put this as

$$bc + bd = b(c + d).$$

In words, b times c plus b times d equals b times (c plus d).

We could have several sets of brackets, such as $(a + b)(c + d)$. Multiplying this out, we see that it equals

$$(a + b)(c + d) = ac + ad + bc + bd.$$

If we put in particular numerical values, then we get the same answer if we work out $a + b$ and $c + d$ and multiply them, or if we work out the four multiplications on the right-hand-side of this equation and add them up. It's quicker to do it the first way.

Brackets also are useful when we are dealing with fractions. Consider the expression

$$\frac{a + b}{c + d}.$$

To write this with a slash, the correct form is $(a + b)/(c + d)$. It could be tempting to write $a + b/c + d$, but that actually means

$$a + \frac{b}{c} + d,$$

which is not at all the same.

This has to be kept in mind when using a calculator. Suppose we want to find (2+3)/(4+6), which obviously is 5/10 or 1/2 = 0.5. If we key it in without brackets, we are likely to get 8.75 which is seriously wrong. If your calculator has brackets, use them in a case like this. Otherwise you could work out the numerator 2+3 and denominator 4+6 separately and then divide them.

In the fraction $(a + b)/(c + d)$, $(a + b)$ is called the *numerator* and $(c + d)$ the *denominator*.

We also have to be careful about the way we may split up a fraction like this. In fact,

$$\frac{a + b}{c + d} = \frac{a}{c + d} + \frac{b}{c + d} = a/(c + d) + b/(c + d).$$

There could be a temptation to think that it equals, for example, $a/c + b/d$ but this is not correct, as you may verify by putting in some numbers.

3. Powers

Integer Powers

If we multiply the same quantity by itself several times then we can express this as a power of that number. For instance,

$$a \times a = a^2,$$

$$a \times a \times a = a^3,$$

$$a \times a \times a \times a = a^4,$$

and so on. Powers combine in a simple way,

$$a^2 \times a^3 = a^{(2+3)} = a^5,$$

or

$$a^3/a^2 = a^{(3-2)} = a^1 = a.$$

We read a^2 in words as 'a squared' and a^3 as 'a cubed'. For a^4 we say 'a to the power of 4' or 'a to the fourth power', and so on for other numbers.

As you see from this example, anything to the power of 1 equals itself. If we have an expression such as a^2/a^2 then this clearly must equal 1 (think of $a = 2$ and $2^2/2^2 = 4/4 = 1$). But also

$$a^2/a^2 = a^{(2-2)} = a^0$$

and in fact any number to the power of zero equals 1.

A special case is that one to any power still equals one, $1^n = 1$.

We can also have negative powers, such as

$$a^2/a^4 = a^{(2-4)} = a^{-2} = 1/a^2,$$

which is 'one over a squared' or 'a to the minus two'.

One over a number, or a number to the power of minus one, $1/a = a^{-1}$, is given a special name. It is the *reciprocal* of the number.

The power to the -1 gives another way to write a fraction. Thus

$$\frac{a}{b} = a/b = ab^{-1}.$$

This is often used with units, such as velocity. Kilometres per second can be written as km/s or as km s^{-1}.

Powers of 10

In Physics and Astronomy we often have to work with numbers that are very big or very small. To avoid writing a lot of figures or decimal places we make a great deal of use of powers of 10. So

$$\begin{aligned} 10^0 &= 1 \\ 10^1 &= 10 \\ 10^2 &= 100 \\ 10^3 &= 1000 \\ 10^4 &= 10000 \end{aligned}$$

and so on; and

$$\begin{aligned} 10^{-1} &= 0.1 \\ 10^{-2} &= 0.01 \\ 10^{-3} &= 0.001 \\ 10^{-4} &= 0.0001 \end{aligned}$$

Fractional Powers and Roots

Suppose $b = a^2$ and we want to find an expression for a. We can do this by *taking the square root* of both sides of this relation, so that

$$a = \sqrt{b}.$$

The square root of b is a number which when multiplied by itself gives b and it is convenient to write this as $b^{1/2}$ so that

$$a^2 = \left(\sqrt{b}\right)^2 = \left(b^{1/2}\right)^2 = b^{1/2} \times b^{1/2} = b^{1/2+1/2} = b.$$

We can take other roots too,

$$\sqrt[3]{b} = b^{1/3},$$

$$\sqrt[4]{b} = b^{1/4},$$

and so on. We may write $b^{1/2}$ as $b^{0.5}$ and $b^{1/4}$ as $b^{0.25}$ but this doesn't work in every case—for example, $b^{1/3} = b^{0.33333...}$ which goes on for ever.

The reciprocal of a root is simply defined, for example

$$\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2} = x^{-0.5}.$$

The integer and fractional powers can be combined in a variety of ways. For example, $a^{5/2}$ is the square root of a^5. It is also the fifth power of the square root of a. So

$$a^{5/2} = \sqrt{(a^5)} = \left(a^5\right)^{\frac{1}{2}} = \left(\sqrt{a}\right)^5 = \left(a^{1/2}\right)^5.$$

Note that to take the power of a power the numbers are to be multiplied, $\frac{1}{2} \times 5 = \frac{5}{2}$. This quantity is *not* the same thing as $a^5 \times a^{1/2}$ which is actually $a^{(5+1/2)}$ or $a^{11/2}$.

Irrational Powers

An *irrational number* is a number that can't be written in terms of integers or fractions. An example is $\pi = 3.14159\ldots$ which extends to an infinite number of digits. Besides powers that can be expressed as integers or fractions, we sometimes come across powers that are irrational. To define what x^y means, if y is an irrational number, requires some more advanced mathematics, related to logarithms, but happily it is possible to work with such quantities without needing to know how they are defined. Any pocket calculator will let you find x^y for any values of x and y.

4. Infinity

Infinity, written as ∞, is bigger than the biggest number you can think of. It is the reciprocal of zero; indeed anything divided by zero gives infinity:

$$\frac{a}{0} = \infty.$$

Infinity doesn't follow normal rules of algebra: if we multiply ∞ by anything, or add anything to it, or take the square root, $\ldots$, we still get ∞. The quantities ∞/∞ or $\infty \times 0$ are mathematically undefined, that is they do not take any particular value.

5. Special Symbols

Equality

In an equation, $a = b$, the two sides have the same value. Mathematicians consider two different types of equation, those which are true whatever the values of the quantities involved, such as

$$a(b + c) = ab + ac,$$

and those which are true only for certain particular values, such as

$$x = a + b.$$

If $a = 2$ and $b = 3$ this is satisfied only by $x = 5$. The first case is called an *identity* and is sometimes written with $\equiv$ instead of $=$,

$$a(b + c) \equiv ab + ac.$$

Sometimes we may want to say that two quantities are *approximately* the same. For instance, the number of seconds in a year can be worked out precisely but in some calculations it is enough to use the approximate value 3×10^7. We could then write

$$1 \text{ year} \sim 3 \times 10^7 \text{ seconds}.$$

The symbols $\sim$, $\approx$ and $\simeq$ all can be used for approximate equality.

If two quantities are not equal we can write it as

$$2 + 2 \neq 5.$$

The symbol $\propto$ is used to indicate that the left hand side is *proportional* to the right hand side, without giving the full relation or other factors that come in. For example, for a planet in orbit around the Sun the gravitational force is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and M and m are the two masses. If we want to emphasise the dependence on the distance r we could write

$$F \propto \frac{1}{r^2},$$

leaving out factors that are for the present unimportant.

To express inequality, we can use $>$ (greater than) or $\geq$ (greater than or possibly equal to) or $\gg$ (much greater than), with corresponding symbols $<$, $\leq$ and $\ll$ for less than. Thus,

$$2 + 2 > 3,$$

$$1 \ll 10^9.$$

Summation and Multiplication
The Greek letter Σ (which is an upper case *sigma* and quite different in appearance from the lower case σ) is used to indicate the sum of a series of terms. Often the starting and finishing values of an index are added as subscript and superscript. So we have

$$\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30.$$

If there are n stars in a cluster, with individual masses m_i, the total mass M of the cluster is

$$M = m_1 + m_2 + m_3 + \ldots + m_n = \sum_{i=1}^{n} m_i.$$

In the same way, Π which is upper case *pi* is used for a product:

$$\prod_{n=1}^{4} n^2 = 1^2 \times 2^2 \times 3^2 \times 4^2 = 1 \times 4 \times 9 \times 16 = 576.$$

6. Working things out

Suppose we have a complicated expression such as

$$(a+b)\left(c - \frac{(d+k)}{e}\right) - (f - gj)(p^{h+k/l} + q)$$

where we are given numerical values for all the symbols. The correct order to work things out is
first, functions
next, expressions in brackets
next, multiplications and divisions
next, additions and subtractions
An example of a function here is taking a power of p. Brackets include the numerator or denominator of a fraction, or an exponent, even if brackets are not explicitly used in the formula. In a complicated expression, with brackets within brackets, we start with what is inside and work out.

Before we start, a word about terminology. In an expression like this, the quantities that are added or subtracted are called *terms*. In a product, the things to be multiplied are called *factors*. It's not uncommon for people to say *terms* when they mean *factors*.

In the first term here, we first work out the bracket $(d + k)$ and then the fraction $(d + k)/e$. We can then work out the two brackets $(a + b)$ and $(c - (d + k)/e)$ and multiply them together.

In the second term, we first find k/l, then $h + k/l$, and then work out the power $p^{h+k/l}$. We then add this to q to get the second factor. For the first factor, first multiply g by j, then subtract the result from f. The two factors can then be multiplied.

Finally, we can subtract the second term from the first one to get our answer.

7. Rearranging Equations

Very often we want to rearrange an equation to get an expression for one or other of the quantities involved. We can change an equation in a variety of ways, so long as we do

exactly the same to both sides of the equation. We could:
multiply or divide by any number or formula;
add or subtract anything;
take the reciprocal;
take any power (e.g., square or take the square root);
take any function (e.g., take 10 to the power of each side);
exchange the two sides of the equation: if $a = b$ then $b = a$.

In a few cases we have to be careful as there can be multiple solutions. In particular, a is a square root of a^2 but so also is $-a$, because $(-a) \times (-a) = a \times a = a^2$. So for the square root of $b = a^2$ we have to say that $b^{1/2} = a$ or $-a$.

Consider the formula for the gravitational force between two objects of mass M and m a distance r apart:

$$F = \frac{GMm}{r^2}.$$

Suppose we want to find an expression for the mass M. We can first multiply both sides by r^2:

$$F \times r^2 = \frac{GMm}{r^2} \times r^2 = GMm.$$

Next, divide through by Gm:

$$\frac{Fr^2}{Gm} = \frac{GMm}{Gm} = M,$$

as we can cancel out the G and the m in the numerator and denominator on the right hand side. Finally, interchange the two sides to obtain

$$M = \frac{Fr^2}{Gm}.$$

Starting from the same formula, let us find an expression for r. We start by inverting both sides (that is, taking the reciprocal)

$$\frac{1}{F} = \frac{r^2}{GMm}$$

and exchanging the two sides

$$\frac{r^2}{GMm} = \frac{1}{F}.$$

Next, multiply both sides by GMm:

$$\frac{r^2}{GMm} GMm = \frac{1}{F} GMm$$

or

$$r^2 = \frac{GMm}{F}.$$

Then take the square root:

$$r = \pm\sqrt{\frac{GMm}{F}}$$

where $\pm$ means *either plus or minus*. In this case we can reject the possible negative root, for the distance between two objects always is a positive number, so

$$r = \sqrt{\frac{GMm}{F}} = \left(\frac{GMm}{F}\right)^{\frac{1}{2}},$$

the last form being more generally used.

As you get more practiced, you'll find that you can do several steps simultaneously or in your head. There are generally different ways to come to the same result.

8. Logarithms

Logarithms lie just beyond what we expect you to know for the Certificate course, but they can make it easier to understand an important astronomical topic, the system of stellar magnitudes, so we include a brief mention here.

Logarithms to base 10

The logarithm of a number is an example of what is called a *function*, which is a prescription for getting another number. In this case, we have two related equations,

$$a = \log b \text{ and } b = 10^a.$$

Logarithms satisfy these relations:

$$\log 1 = 0$$

$$\log 10 = 1$$

$$\log(a^c) = c \log a$$

$$\log(ab) = \log a + \log b$$

$$\log\left(\frac{a}{b}\right) = \log a - \log b.$$

These formulae are for what is called *base 10* and are sometimes written in the form $\log_{10} a$. Mathematically, logarithms can use any number as the base, and for example for a base p the defining equations become

$$a = \log_p b \text{ and } b = p^a.$$

Such general forms are met in mathematics; in astronomy base 10 is used a lot and we usually just write log and not $\log_{10}$.

Before electronic calculators were available, log tables were used as an important aid to calculation. Suppose we want to multiply two numbers a and b. Then from the tables we find the logarithms $\log a$ and $\log b$ and add them to get

$$\log(ab) = \log a + \log b$$

Then from the tables we find the number whose logarithm this is, that is ab. This procedure is much faster than doing a long multiplication by hand.

Natural Logarithms

There is a number $e = 2.718\ldots$ that plays an important part in the equations of theoretical physics. It is, like π, an irrational number and you may find it on your calculator. Logarithms to base e are important in theory. They are called *natural logarithms*; they can be written $\log_e$ but instead we often use ln. You might sometimes see this in equations.

Note that this number e has nothing to do with the charge on an electron, for which we generally use the same symbol.

9. Trigonometry

Trigonometry is a branch of mathematics that lies decidedly beyond what is needed in the Certificate course, and you will not have to work with it, but you might see some of the functions in equations and wonder what they are.

In trigonometry we are concerned with the properties of angles. With 360 degrees (360°) in a complete circle, 90° is a right angle. The trigonometric functions most commonly met are the *sine* and the *cosine* of an angle. If we have an angle θ (Greek *theta*), these are written as $\sin\theta$ and $\cos\theta$. They have particular values:

$$\sin 0 = 0; \quad \sin 90^\circ = 1; \quad \cos 0 = 1; \quad \cos 90^\circ = 0$$

These are oscillating functions—the values for θ in the range from 0° to 360° are repeated from 360° to 720° and so on. We talk about a *sine wave* or some function varying *sinusoidally*.

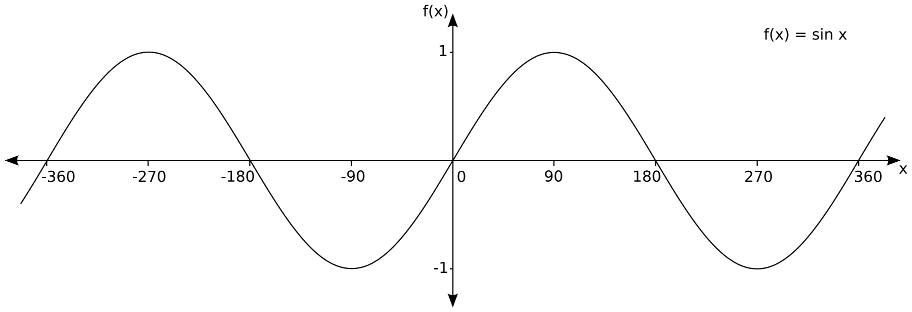

Figure 1: An example of a sine wave.

In mathematical work, angles often are expressed not in degrees but in *radians*. The radian is in some ways a natural unit of angle, whereas the degree is totally artificial. The relation is that $360^\circ = 2\pi$ radians. Then $180^\circ = \pi$ radians and a right angle is $90^\circ = \pi/2$ radians.

If you should have to calculate the sin or cosine of an angle, note that your calculator will be able to do this either using degrees or using radians. Make sure that it is set to the units that you are using. Otherwise you'll get a wrong answer.

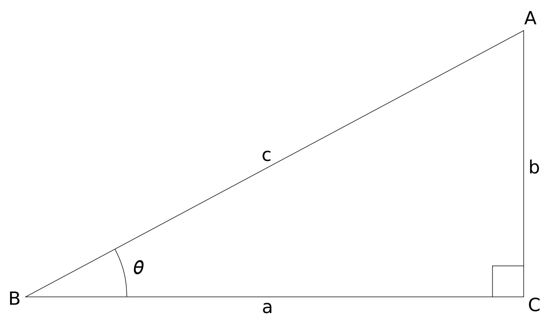

Figure 2: A triangle containing a right angle.

The original definitions of the trigonometric functions were within the geometry of a triangle containing a right angle, that is an angle which equals 90°. Within the triangle shown, if θ is one of the other angles then

$$\sin\theta = b/c$$

and

$$\cos\theta = a/c.$$

Because these are ratios, it doesn't matter what units of length are used for a, b and c, whether cm or Mpc (so long as they all are the same)—the ratios are unaltered if we change from one unit to another.

These give the values above if we consider what happens as θ becomes very small—then b tends towards zero and c tends towards a, so in this limit

$$\sin 0 = b/c = 0/c = 0; \quad \cos 0 = a/c = a/a = 1.$$

As θ tends towards $90°$ then a tends towards zero and c tends towards b so that

$$\sin 90 = b/c = b/b = 1; \quad \cos 90 = a/c = 0/c = 0.$$

Within the triangle, θ can't be larger than $90°$, but the definitions can be extended to apply to any value of θ including negative values.

This has an application in the projection of a distance in a different direction. From the definition

$$\cos\theta = a/c$$

and multiplying through by c we have that

$$a = c\cos\theta$$

so the projection of c in direction θ is $c\cos\theta$.

In astronomy we frequently come across projection effects.